

EXISTENCE OF THE DIRECTIONAL TANGENT CONE TO A POSITIVE CURRENT

HAITHAM HAWARI, JAWHAR HBIL, AND NOUREDDINE GHILOUFI

ABSTRACT. In this paper, we start by proving the existence of the strict transform of a positive current T as soon as its j^{th} currents, T_j , are plurisubharmonics or plurisuperharmonics. Then, with a suitable condition on T_j , we show the existence of the directional tangent cone to T . In the particular case, when T is closed, we give a second condition independent to the previous one.

Existence du cône tangent directionnel à un courant positif.

RÉSUMÉ. Dans la première partie de cet article, on montre l'existence du relèvement d'un courant positif T , par un clatement de centre lisse, dès que ses $j^{ème}$ courants T_j soient plurisousharmoniques ou plurisurharmoniques. Ensuite, avec une condition supplémentaire sur les courants T_j , on prouve l'existence du cône tangent directionnel à T . Dans le cas particulier où T est fermé, une deuxième condition indépendante de la première sera donnée.

1. INTRODUCTION

We are interested to the problem of the existence of the strict transforms of positive currents and the problem of the existence of the directional tangent cones to positive currents. These two problems are closely related, especially in the case of positive closed currents.

For the second problem, it is well known that the tangent cone to the current of integration, over an analytic set, exists and coincides with the current of integration over the geometric tangent cone. This result had been proved by Thie in 1967 and King in 1971. In the other side, Kiselman [8] in 1988, gave an example of a positive closed current of bidegree $(1, 1)$ which has not a tangent cone. It is therefore natural to find a sufficient condition for its existence. Based on the construction of Kiselman which use essentially the projective mass of the current, Blel, Demailly and Mouzali [2] have given two independent conditions where each one ensure the existence of the tangent cone to a positive closed current. Recently, Haggui [7] has shown that one of these conditions remains sufficient even for the case of positive plurisubharmonic currents. This result has been found and generalized by Ghiloufi

2000 *Mathematics Subject Classification.* 32U25; 32U40.

Key words and phrases. Lelong number, positive plurisubharmonic current, Tangent cone.

and Dabbek [4] in the case of a positive plurisubharmonic (psh) or plurisuperharmonic (prh) current.

For the first problem, Giret [6] has given, in 1998, some estimates of the projective mass of a positive closed current which allows him to give a sufficient condition for the existence of a strict transform of a positive current. Using Raby's and Babouche's works, [9, 1] on the problem of restriction of a positive closed current on hypersurfaces, Giret gave a link between the existence of directional tangent cone and the strict transform of this current.

The basic purpose of this work is to study the existence of the directional tangent cone and the strict transform of positive psh or prh currents.

In the hole of this paper, we consider $n, m, p \in \mathbb{N}$ such that $0 < p < n$. We use (z, t) to indicate an element of $\mathbb{C}^N := \mathbb{C}^n \times \mathbb{C}^m$ containing 0. Let $\Omega := \Omega_1 \times \Omega_2$ be an open subset of \mathbb{C}^N and B be an open subset of Ω_2 . We assume that there exists $R > 0$ such that $\{z \in \mathbb{C}^n; |z| < R\} \times B$ is relatively compact in Ω . For $0 < r < R$ and $r_1 < r_2 < R$, we set: $\mathbb{B}_n(r) = \{z \in \Omega_1; |z| < r\}$, $\mathbb{B}_n(r_1, r_2) = \{z \in \Omega_1; r_1 \leq |z| < r_2\}$ and $\mathbb{B}_m(r) = \{t \in \Omega_2; |t| < r\}$.

To simplify the notation, we set $\beta_z := dd^c|z|^2$, $\beta_t := dd^c|t|^2$ and $\alpha_z := dd^c \log |z|^2$.

Let T be a positive current of bidegree (p, p) on Ω . The directional Lelong number of T with respect to B is defined, when it exists, as the limit, $\nu(T, B) := \lim_{r \rightarrow 0^+} \nu(T, B, r)$ where $\nu(T, B, \cdot)$ is the function defined by

$$\nu(T, B, r) := \frac{1}{r^{2(n-p)}} \int_{\mathbb{B}_n(r) \times B} T \wedge \beta_z^{n-p} \wedge \beta_t^m.$$

In the particular case $m = 0$, we omit B in previous notation and to make a distinction, we note $\nu_T(r)$, $\nu_T(0)$ to indication respectively classical projective mass and Lelong number of T at 0.

The following lemma will be useful:

Lemma 1. (*Lelong-Jensen formula*) (See [3]) *Let S be a positive psh or prh current of bidimension (q, q) on the ball $\mathbb{B}_n(R)$ of \mathbb{C}^n with $0 < q < n$. Then for any $0 < r_1 < r_2 < R$,*

$$\begin{aligned} & \nu_S(r_2) - \nu_S(r_1) \\ &:= \frac{1}{r_2^{2q}} \int_{\mathbb{B}_n(r_2)} S \wedge \beta_z^q - \frac{1}{r_1^{2q}} \int_{\mathbb{B}_n(r_1)} S \wedge \beta_z^q \\ &= \int_{\mathbb{B}_n(r_1, r_2)} S \wedge \alpha_z^q + \int_{r_1}^{r_2} \left(\frac{1}{s^{2q}} - \frac{1}{r_2^{2q}} \right) s ds \int_{\mathbb{B}_n(s)} dd^c S \wedge \beta_z^{q-1} \\ & \quad + \left(\frac{1}{r_1^{2q}} - \frac{1}{r_2^{2q}} \right) \int_0^{r_1} s ds \int_{\mathbb{B}_n(s)} dd^c S \wedge \beta_z^{q-1}. \end{aligned}$$

According to Lemma 1, if S is a positive psh current then ν_S is a non-negative increasing function on $]0, R[$, therefore the Lelong number $\nu_S(0) := \lim_{r \rightarrow 0^+} \nu_S(r)$ of S at 0 exists.

For a positive prh current one has the following result:

Proposition 1. (See [5]) *Let S be a positive prh current of bidimension (q, q) on the ball $\mathbb{B}_n(R)$ of \mathbb{C}^n , $0 < q < n$. We assume that S satisfies the condition (C_0) given by*

$$(C_0) : \quad \int_0^{r_0} \frac{\nu_{dd^c S}(s)}{s} ds > -\infty$$

for $0 < r_0 < R$. Then, the Lelong number $\nu_S(0)$ of S at 0 exists.

Proof. For $0 < r < R$, we set

$$\Lambda_S(r) = \nu_S(r) + \int_0^r \left(\frac{s^{2q}}{r^{2q}} - 1 \right) \frac{\nu_{dd^c S}(s)}{s} ds.$$

The function Λ_S is well defined and non-negative on $]0, R[$. In addition, by the Lelong-Jensen formula, it's easy to show that this function is increasing on $]0, R[$, hence the existence of the limit $\ell := \lim_{r \rightarrow 0^+} \Lambda_S(r)$. Condition (C_0) and the fact that $(s^q/r^q - 1)$ is uniformly bounded give

$$\lim_{r \rightarrow 0^+} \int_0^r \left(\frac{s^{2q}}{r^{2q}} - 1 \right) \frac{\nu_{dd^c S}(s)}{s} ds = 0.$$

So, $\ell = \lim_{r \rightarrow 0^+} \Lambda_S(r) = \lim_{r \rightarrow 0^+} \nu_S(r) = \nu_S(0)$. \square

We end this part by recalling the Demailly's Inequality which will be useful in the proofs of various results in this paper:

If

$$S = 2^{-q} i^{q^2} \sum_{|I|=|J|=q} S_{I,J} dw_I \wedge d\bar{w}_J$$

is a positive (q, q) -current, then for all $(\lambda_1, \dots, \lambda_n) \in]0, +\infty[^n$ we have

$$\lambda_I \lambda_J |S_{I,J}| \leq 2^q \sum_{M \in \mathcal{M}_{I,J}} \lambda_M S_{M,M}$$

where $\lambda_I = \lambda_{i_1} \dots \lambda_{i_q}$ when $I = (i_1, \dots, i_q)$ and the sum is taken over the set of q -index $\mathcal{M}_{I,J} = \{M; |M| = q, I \cap J \subset M \subset I \cup J\}$.

2. STRICT TRANSFORM OF A POSITIVE CURRENT

Let $\mathbb{C}^N[\{0\} \times \mathbb{C}^m] := \{(z, \mathbf{g}, t) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \times \mathbb{C}^m; z \in \mathbf{g}\}$ be the blow-up of $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ with center $L := \{0\} \times \mathbb{C}^m$ and $\pi : \mathbb{C}^N[\{0\} \times \mathbb{C}^m] \rightarrow \mathbb{C}^N$ be the canonical projection defined by $\pi(z, \mathbf{g}, t) = (z, t)$. Let T be a positive current on \mathbb{C}^N . We try to find a positive current \widehat{T} on $\mathbb{C}^N[\{0\} \times \mathbb{C}^m]$ such that $\pi_* \widehat{T} = T$; a such current, if it exists, will be called the strict transform of T . If $m = 0$, it was shown (See [4]) that this strict transform exists in the

case of positive psh currents or positive prh currents satisfying Condition (C_0) . That's why we will consider the case $m \neq 0$. In this case Kiselman [8] had shown that this strict transform may not exist. We try to find some sufficient conditions on T to ensure the existence of its strict transform. We define the current T_j as $T_j := \int_{\mathbb{B}_m(\sigma)} T \wedge \beta_t^{m-j}$ for any integer $0 \leq j \leq m$. The current T_j is positive of bidimension $(n-p+j, n-p+j)$ on \mathbb{C}^n where (p, p) is the bidegree of T . We assume that the currents T_j are psh or prh satisfying Condition (C_0) , so the Lelong number $\nu_{T_j}(0)$ of T_j at 0 exists; this number will be called the j^{th} Lelong number of T at 0.

Theorem 1. *Let T be as above. Assume that for $j \in J_1$ (resp. $j \in J_2$), the current $T_j := \int_{\mathbb{B}_m(\sigma)} T \wedge \beta_t^j$ is plurisubharmonic (resp. plurisuperharmonic satisfying Condition (C_0)) for every $\sigma > 0$ where $J_1 \cup J_2 = \{0, \dots, m\}$. Then the strict transform of T exists. furthermore, there exists a constant $c > 0$ such that for every $0 < r < R$, one has*

$$\begin{aligned} & \|\pi^*(T|_{\Omega \setminus L})\|(\pi^{-1}[(\mathbb{B}_n(r) \setminus \{0\}) \times \mathbb{B}_m(\sigma)]) \\ & \leq c \sum_{j=0}^m (|\nu_{T_j}(r) - \nu_{T_j}(0)| + C_1(r)\nu_{T_j}(r)) - C_2(r) \sum_{j \in J_2} \int_0^r \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds. \end{aligned}$$

where $C_1(r) = \sum_{k=1}^{N-p} r^{2k}$ and $C_2(r) = \sum_{k=0}^{N-p} r^{2k}$

This Theorem generalizes a result of Giret where he considered the case $J_2 = \emptyset$ (all T_j are psh).

Proof. Let ω be the Kähler form of $\mathbb{C}^N[\{0\} \times \mathbb{C}^m]$. We have:

$$\begin{aligned} (\pi_* \omega)^{N-p} &= (\beta_t + \beta_z + \alpha_z)^{N-p} \\ &= \sum_{k=0}^{N-p} C_{N-p}^k (\beta_t + \beta_z)^k \wedge \alpha_z^{N-p-k} \\ &= \sum_{k=0}^{N-p} C_{N-p}^k \left(\sum_{j=0}^{\min(m,k)} C_k^j \beta_t^j \wedge \beta_z^{k-j} \right) \wedge \alpha_z^{N-p-k}. \end{aligned}$$

It is therefore sufficient to control the integrals:

$$\int_{\mathbb{B}_n(\varepsilon, r) \times \mathbb{B}_m(\sigma)} T \wedge \alpha_z^{N-p-k} \wedge \beta_z^{k-j} \wedge \beta_t^j$$

for any integers $0 \leq k \leq N-p$, $0 \leq j \leq \min(k, m)$ and $0 < \varepsilon < r$.

For $0 \leq k \leq N-p$ and $0 \leq j \leq \min(k, m)$, the Lelong-Jensen formula applied to $T_{m-j} \wedge \beta_z^{k-j}$ gives:

$$\begin{aligned}
(2.1) \quad & \int_{\mathbb{B}_n(\epsilon, r) \times \mathbb{B}_m(\sigma)} T \wedge \alpha_z^{N-p-k} \wedge \beta_z^{k-j} \wedge \beta_t^j \\
= & \frac{1}{r^{2(N-p-k)}} \int_{\mathbb{B}_n(r) \times \mathbb{B}_m(\sigma)} T \wedge \beta_z^{N-p-j} \wedge \beta_t^j \\
& - \frac{1}{\epsilon^{2(N-p-k)}} \int_{\mathbb{B}_n(\epsilon) \times \mathbb{B}_m(\sigma)} T \wedge \beta_z^{N-p-j} \wedge \beta_t^j \\
& - \int_0^r \left(\frac{1}{s^{2(j-k)}} - \frac{s^{2(N-p-j)}}{r^{2(N-p-k)}} \right) \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds \\
& + \int_0^\epsilon \left(\frac{1}{s^{2(j-k)}} - \frac{s^{2(N-p-j)}}{\epsilon^{2(N-p-k)}} \right) \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds \\
= & r^{2(k-j)} \nu_{T_{m-j}}(r) - \int_0^r s^{2(k-j)} \left(1 - \frac{s^{2(N-p-k)}}{r^{2(N-p-k)}} \right) \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds \\
& - \epsilon^{2(k-j)} \nu_{T_{m-j}}(\epsilon) + \int_0^\epsilon s^{2(k-j)} \left(1 - \frac{s^{2(N-p-k)}}{\epsilon^{2(N-p-k)}} \right) \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds.
\end{aligned}$$

If $j \in J_1$, Equality 2.1 gives

$$\begin{aligned}
& \int_{\mathbb{B}_n(\epsilon, r) \times \mathbb{B}_m(\sigma)} T \wedge \alpha_z^{N-p-k} \wedge \beta_z^{k-j} \wedge \beta_t^j \\
\leq & r^{2(k-j)} \nu_{T_{m-j}}(r) - \epsilon^{2(k-j)} \nu_{T_{m-j}}(\epsilon) \\
& + \int_0^\epsilon s^{2(k-j)} \left(1 - \frac{s^{2(N-p-k)}}{\epsilon^{2(N-p-k)}} \right) \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds.
\end{aligned}$$

If $j \in J_2$, Equality 2.1 gives

$$\begin{aligned}
& \int_{\mathbb{B}_n(\epsilon, r) \times \mathbb{B}_m(\sigma)} T \wedge \alpha_z^{N-p-k} \wedge \beta_z^{k-j} \wedge \beta_t^j \\
\leq & r^{2(k-j)} \nu_{T_{m-j}}(r) - r^{2(k-j)} \int_0^r \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds \\
& - \epsilon^{2(k-j)} \nu_{T_{m-j}}(\epsilon) + \int_0^\epsilon s^{2(k-j)} \left(1 - \frac{s^{2(N-p-k)}}{\epsilon^{2(N-p-k)}} \right) \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds.
\end{aligned}$$

Taking the limit when ϵ goes to 0, we deduce that the current $\pi^*(T|_{\Omega \setminus L})$ has a locally finite mass in a neighborhood of points of $E = \pi^{-1}(L)$. More precisely, there exists a constant c such that

$$\begin{aligned}
& \|\pi^*(T|_{\Omega \setminus L})\|(\pi^{-1}\{(\mathbb{B}_n(r) \setminus \{0\}) \times \mathbb{B}_m(\sigma)\}) \\
\leq & c \sum_{j=0}^m (|\nu_{T_{m-j}}(r) - \nu_{T_{m-j}}(0)| + C_1(r) \nu_{T_{m-j}}(r)) \\
& - \sum_{j \in J_2} C_2(r) \int_0^r \frac{\nu_{dd^c T_{m-j}}(s)}{s} ds.
\end{aligned}$$

Then the trivial extension \widehat{T} of $\pi^*(T|_{\Omega \setminus L})$ by zero over $\pi^{-1}(L)$ exists. The current $\pi_*\widehat{T} - T$ is \mathbb{C} -flat of bidimension $(n + m - p, n + m - p)$ on \mathbb{C}^N supported by $\{0\} \times \mathbb{C}^m$, hence it vanishes by the support theorem. \square

Remark 1. Let T be a positive current with support in the stripe $\Omega_1 \times \mathbb{B}_m(\sigma_0)$.

- (1) If T is plurisubharmonic then T has a positive strict transform.
- (2) If T is plurisuperharmonic and $T_j = \int_{\mathbb{B}_m(\sigma_0)} T \wedge \beta_t^{m-j}$ satisfy Condition (C_0) , $0 \leq j \leq m$, then T has a positive strict transform.

3. DIRECTIONAL TANGENT CONE TO A POSITIVE CURRENT

In this section we denote by $T_j := \int_B T \wedge \beta_t^{m-j}$ where B is an open set of \mathbb{C}^m .

Definition 1. Let T be a positive current on an open set Ω of $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$. The directional tangent cone to T is the weak limit, if it exists, of the family $(h_a^*T)_{a \in \mathbb{C}}$ as $a \rightarrow 0$ where h_a is the directional dilatation on \mathbb{C}^N defined by $h_a(z, t) = (az, t)$.

The main result of this paper is the following:

Theorem 2. *Let T be a positive current of bidegree (p, p) on Ω . We assume that for any integer $0 \leq j \leq \min(m, p)$, the current T_j is psh (resp. prh) satisfying Condition (C_0) and*

$$\int_0^{r_0} \frac{|\nu_{T_j}(r) - \nu_{T_j}(0)|}{r} dr < +\infty.$$

Then the directional tangent cone to T exists.

We start by giving some results useful for the proof of this theorem.

Remark 2. Let T be as above and B an open set of \mathbb{C}^m , then for all integer $0 \leq j \leq m$, the current T_j is positive, of bidimension $(n - p + j, n - p + j)$ on \mathbb{C}^n and one has:

$$h_a^*T_j = (h_a^*T)_j.$$

In fact, we can assume, without loss of generality, that T is \mathcal{C}^∞ . Then

$$h_a^* \int_B T \wedge \beta_t^{m-j} = h_a^* p_{1*} \left(T \wedge \beta_t^{m-j} |_{\mathbb{C}^n \times B} \right)$$

where

$$\begin{aligned} p_1 : \quad \mathbb{C}^n \times \mathbb{C}^m &\longrightarrow \mathbb{C}^n \times \mathbb{C}^m \\ (z, t) &\longmapsto (z, 0). \end{aligned}$$

One has

$$\begin{aligned}
h_a^* p_{1\star} \left(T \wedge \beta_t^{m-j} |_{\mathbb{C}^n \times B} \right) &= (h_{\frac{1}{a}})_* p_{1\star} \left(T \wedge \beta_t^{m-j} |_{\mathbb{C}^n \times B} \right) \\
&= \left(p_1 \circ h_{\frac{1}{a}} \right)_* \left(T \wedge \beta_t^{m-j} |_{\mathbb{C}^n \times B} \right) \\
&= \left(h_{\frac{1}{a}} \circ p_1 \right)_* \left(T \wedge \beta_t^{m-j} |_{\mathbb{C}^n \times B} \right) \\
&= p_{1\star} \left(h_a^* T \wedge \beta_t^{m-j} |_{\mathbb{C}^n \times B} \right) \\
&= \int_B h_a^* T \wedge \beta_t^{m-j}.
\end{aligned}$$

□

Lemma 2. *Let T be a positive current of bidimension (p, p) on an open set Ω of $\mathbb{C}^n \times \mathbb{C}^m$. We assume that the current T_j is psh for $j \in J_1$ and prh satisfying Condition (C_0) for $j \in J_2$ where $J_1 \cup J_2 = \{0, \dots, \min(m, p)\}$. Then there exists $c > 0$ such that*

$$\begin{aligned}
(3.1) \quad &\int_{\mathbb{B}_n(r) \times B} h_a^* T \wedge (\beta_z + \beta_t)^{N-p} \\
&\leq C r^{2(n-p)} \left(\sum_{j=0}^{\min(m,p)} \nu_{T_j}(r_0) + \sum_{j \in J_2} \int_0^{r_0} \left(\frac{s^{2(n-p+j-1)}}{r_0^{2(n-p+j-1)}} - 1 \right) \frac{\nu_{dd^c T_j}(s)}{s} ds \right)
\end{aligned}$$

for $|a|r \leq r_0$.

Proof. For $r > 0$, we have

$$\int_{\mathbb{B}_n(r) \times B} h_a^* T \wedge (\beta_z + \beta_t)^{N-p} = \sum_{j=0}^{\min(m,p)} C_{N-p}^j \int_{\mathbb{B}_n(r) \times B} h_a^* T \wedge \beta_z^{n-p+j} \wedge \beta_t^{m-j}.$$

Furthermore, if $j \in J_1$ then $\nu_{h_a^* T_j}(r) = \nu_{T_j}(|a|r) \leq \nu_{T_j}(r_0)$ for $r \leq \frac{r_0}{|a|} \leq \min(1, \frac{r_0}{|a|})$. It follows that:

$$\int_{\mathbb{B}_n(r) \times B} h_a^* T \wedge \beta_z^{n-p+j} \wedge \beta_t^{m-j} \leq r^{2(n-p+j)} \nu_{T_j}(r_0) \leq r^{2(n-p)} \nu_{T_j}(r_0).$$

If $j \in J_2$ then using the proof of Proposition 1, we have

$$\frac{1}{r^{2(n-p+j)}} \int_{\mathbb{B}_n(r) \times B} h_a^* T \wedge \beta_z^{n-p+j} \wedge \beta_t^{m-j} \leq \Lambda_{h_a^* T_j}(r) \leq \Lambda_{T_j}(r_0).$$

It follows that

$$\int_{\mathbb{B}_n(r) \times B} h_a^* T \wedge \beta_z^{n-p+j} \wedge \beta_t^{m-j} \leq r^{2(n-p)} \Lambda_{T_j}(r_0).$$

The result follows by summing on j from 0 to $\min(m, p)$. □

According to the previous lemma the mass $h_a^* T$ converges in a neighborhood of $(0, t)$, then the sequence $h_a^* T$ converges on \mathbb{C}^N if and only if it converges on a neighborhood of (z_0, t) where $z_0 \in \mathbb{C}^n \setminus \{0\}$.

Using a dilatation and changement of coordinates if necessary, we can assume that $(z_0, t) = (0, \dots, z_n^0, t)$ where $\frac{1}{2} < |z_n^0| < 1$. We use the projective coordinates:

$$w_1 = \frac{z_1}{z_n}, \dots, w_n = z_n$$

$$T = \sum_{I,J,K,L} T_{IJ,KL} dw_I \wedge d\bar{w}_J \wedge dt_K \wedge d\bar{t}_L$$

h_a will be written as: $h_a : (w', w_n, t) \mapsto (w', aw_n, t)$. We check that the coefficients $T_{IJ,KL}^a$ of $h_a^* T$ are given by:

$$T_{IJ,KL}^a = \begin{cases} T_{IJ,KL}(w', w_n, t) & \text{if } n \notin I \text{ and } n \notin J \\ a T_{IJ,KL}(w', w_n, t) & \text{if } n \in I \text{ and } n \notin J \\ \bar{a} T_{IJ,KL}(w', w_n, t) & \text{if } n \notin I \text{ and } n \in J \\ |a|^2 T_{IJ,KL}(w', w_n, t) & \text{if } n \in I \text{ and } n \in J \end{cases}$$

Lemma 3. *Let U be a neighborhood of $z_0 \in \mathbb{C}^n$ such that $U \times B \subset \mathbb{B}_n(\frac{1}{2}, 1) \times B$. If we note, for every integer $0 \leq j \leq m$, by*

$$\gamma_{T_j}(r) = \nu_{T_j}(r) - \nu_{T_j}(r/2) \quad \text{and} \quad \gamma_{dd^c T_j}(r) = \nu_{dd^c T_j}(r) - \nu_{dd^c T_j}(r/2)$$

then for $|a| < r_0$, $r_0 < R$, there exists $C_1, C_2, C_3 > 0$ such that the measure $T_{IJ,KL}^a$ satisfies the following estimates:

$$\int_{U \times B} |T_{IJ,KL}^a| \leq \begin{cases} C_1 \\ C_2 \sum_{j=0}^m \gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|) & \text{if } n \in I \text{ and } n \in J \\ C_3 \left(\sum_{j=0}^m \gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|) \right)^{\frac{1}{2}} & \text{if } n \in I \text{ or } n \in J \end{cases}$$

Proof.

- The set \bar{U} is compact and the positive form β is \mathcal{C}^∞ , so one has: $\beta \geq C_4 dd^c |w|^2$ on U . The inequality (3.1), with $r = 1$, implies: $\int_{U \times B} T_{M,M}^a \leq C_5$. The Demailly's inequality with the choice $\lambda_1 = \dots = \lambda_N = 1$, gives

$$\int_{U \times B} |T_{IJ,KL}^a| \leq C_6 \sum_{M \in \mathcal{M}_{IJ,KL}} \int_{U \times B} T_{M,M}^a \leq C_1.$$

The first estimate of the lemma is proved.

- To proof the second estimate, we observe that $\alpha \geq C_7 \beta'$ on U with $\beta' = dd^c |w'|^2$. In fact, $\alpha = dd^c \log(1 + |w'|^2) \geq \frac{1}{(1 + |w'|^2)^2} \beta' \geq \frac{1}{4} \beta'$

on U . It follows that:

$$\begin{aligned}
\int_{U \times B} \sum_{n \in M} T_{M,M}^a &= \int_{U \times B} h_a^* T \wedge (dd^c |w'|^2 + \beta_t)^{N-p} \\
&\leq C_8 \sum_{j=0}^m \int_{U \times B} h_a^* T \wedge \beta_t^{m-j} \wedge \alpha^{n-p+j} \\
&\leq C_8 \sum_{j=0}^m \int_{\mathbb{B}_n(1/2,1) \times B} h_a^* T \wedge \beta_t^{m-j} \wedge \alpha^{n-p+j}.
\end{aligned}$$

Using Lelong-Jensen formula for $T_j = \int_B T \wedge \beta_t^{m-j}$, $r_2 = 1$ and $r_1 = 1/2$, one has:

$$\begin{aligned}
&\int_{\mathbb{B}_n(1/2,1) \times B} h_a^* T \wedge \beta_t^{m-j} \wedge \alpha^{n-p+j} \\
&\leq \nu_{T_j}(|a|) - \nu_{T_j}(|a|/2) - \int_{\frac{1}{2}}^1 \left(\frac{1}{t^{2p}} - 1 \right) t^{2p-1} \nu_{dd^c(h_a^* T_j)}(t) dt \\
&\quad - \left(\frac{1}{2^{2p}} - 1 \right) \int_0^{\frac{1}{2}} t^{2p-1} \nu_{dd^c(h_a^* T_j)}(t) dt \\
&\leq \nu_{T_j}(|a|) - \nu_{T_j}(|a|/2) - \int_{\frac{1}{2}}^1 \frac{\nu_{dd^c T_j}(|a|t)}{t} dt + \int_0^1 t^{2p-1} \nu_{dd^c T_j}(|a|t) dt \\
&\leq (\nu_{T_j}(|a|) - \nu_{T_j}(|a|/2)) + C_9(\nu_{dd^c T_j}(|a|) - \nu_{dd^c T_j}(|a|/2)) \\
&\leq C_8 \gamma_{T_j}(|a|) + C_9 \gamma_{dd^c T_j}(|a|).
\end{aligned}$$

Hence

$$\int_{U \times B} \sum_{n \in M} T_{M,M}^a \leq \sum_{j=0}^m C_8 \gamma_{T_j}(|a|) + C_9 \gamma_{dd^c T_j}(|a|).$$

The second estimate is proved for $n \in M$.

In the general case, $I, J \ni n$, using the Demailly's inequality for $\lambda_1 = \dots = \lambda_N = 1$, we obtain:

$$\int_{U \times B} |T_{IJ,KL}^a| \leq C_{10} \sum_{M \in \mathcal{M}_{IJ,KL}} \int_{U \times B} T_{M,M}^a \leq C_2 \sum_{j=0}^m (\gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|))$$

Hence we get The second estimate.

- For the third estimation, it suffices to assume that $n \in I$ and $n \notin J$. Using the Demailly's inequality again with $\lambda_k = 1$ for every $k \neq n$

and $\lambda_n > 0$, one has:

$$\begin{aligned}
& \lambda_n \int_{U \times B} |T_{IJ,KL}^a| \\
& \leq C_{11} \int_{U \times B} \left(\sum_{n \notin M \in \mathcal{M}_{IJ,KL}} T_{M,M}^a + \lambda_n^2 \sum_{n \in M \in \mathcal{M}_{IJ,KL}} T_{M,M}^a \right) \\
& \leq C_{12} + C_{13} \lambda_n^2 \sum_{j=0}^m (\gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|)).
\end{aligned}$$

The third estimate can be deduced by choosing

$$\lambda_n = \frac{1}{\left(\sum_{j=0}^m \gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|) \right)^{\frac{1}{2}}}.$$

□

Lemma 4. (See [2]) *Let f be a \mathcal{C}^2 function defined on the pointed disc $\mathbb{D}^*(0, r_0) \subset \mathbb{C}$. We assume that f is bounded and there exists a measurable function $u :]0, r_0] \rightarrow \mathbb{R}^+$ where $\int_0^{r_0} r |\log r| u(r) dr < +\infty$ and $|\Delta f(a)| \leq u(|a|)$. Then $f(a)$ has a limit when a goes to 0.*

3.1. Proof of Theorem 2. The proof of the main theorem will be done in the case of positive psh currents, the case of positive prh currents is similar with some simple modifications, that's why we assume that T is positive psh. The lemma 3 shows that the mass of $T_{IJ,KL}^a$ goes to 0 when $n \in I$ or $n \in J$.

For $n \notin I = \{i_1, \dots, i_q\}$ and $n \notin J = \{j_1, \dots, j_{q'}\}$, let $\varphi \in \mathcal{D}(U \times B)$. We set

$$\begin{aligned}
f_{IJ,KL}(a) &= \int_{U \times B} T_{IJ,KL}^a(w, t) \varphi(w, t) d\tau(w, t) \\
&= \int_{U \times B} T_{IJ,KL}(w', aw_n, t) \varphi(w, t) d\tau(w, t).
\end{aligned}$$

The function $f_{IJ,KL}$ is \mathcal{C}^∞ on $\mathbb{D}^*(0, R) := \{a \in \mathbb{C}; 0 < |a| < R\}$ and bounded in a neighborhood of 0. We have:

$$\frac{\partial^2 f_{IJ,KL}}{\partial a \partial \bar{a}}(a) = \int_{U \times B} |w_n|^2 \frac{\partial^2 T_{IJ,KL}}{\partial w_n \partial \bar{w}_n}(w', w_n, t) \varphi(w, t) d\tau(w, t).$$

We observe that the coefficients $dw_{I \cup \{n\}} \wedge d\bar{w}_{J \cup \{n\}} \wedge dt_K \wedge d\bar{t}_J$ in the expression of $dd^c T$ is

$$\begin{aligned}
& (dd^c T)_{I \cup \{n\} J \cup \{n\}, KL} \\
= & (-1)^{q'} \frac{\partial^2 T_{IJ, KL}}{\partial w_n \partial \bar{w}_n} + \sum_{k=1}^q \sum_{s=1}^{q'} (-1)^{q+s+k} \frac{\partial^2 T_{I(k)J(s), KL}}{\partial w_{i_k} \partial \bar{w}_{j_s}} \\
& + \sum_{s=1}^{q'} (-1)^{s-1} \frac{\partial^2 T_{IJ(s), KL}}{\partial w_n \partial \bar{w}_{j_s}} + \sum_{k=1}^q (-1)^{k-1} \frac{\partial^2 T_{I(k)J, KL}}{\partial w_{i_k} \partial \bar{w}_n} \\
& + \sum_{e=1}^{p-q} \sum_{e'=1}^{p-q'} (-1)^{q'+q+e'} \frac{\partial^2 T_{I \cup \{n\} J \cup \{n\}, K_e L_{e'}}}{\partial t_{i_e} \partial \bar{t}_{j_{e'}}} \\
& + \sum_{e'=1}^{p-q'} (-1)^{e'-q'} \frac{\partial^2 T_{IJ \cup \{n\}, KL_{e'}}}{\partial w_n \partial \bar{t}_{j_{e'}}} + \sum_{e=1}^{p-q} (-1)^e \frac{\partial^2 T_{I \cup \{n\} J, K_e L}}{\partial t_{i_e} \partial \bar{w}_n}.
\end{aligned}$$

with $I(k) = I \setminus \{i_k\} \cup \{n\}$, $J(s) = J \setminus \{j_s\} \cup \{n\}$, $K_e = K \setminus \{i_e\}$ et $L_{e'} = L \setminus \{j_{e'}\}$. The previous equality gives:

$$\begin{aligned}
& \frac{\partial^2 f_{IJ, KL}}{\partial a \partial \bar{a}}(a) \\
= & (-1)^{q'} \int_{U \times B} \frac{|w_n|^2}{|a|^2} (dd^c T)_{I \cup \{n\} J \cup \{n\}, KL}^a \varphi(w, t) d\tau(w, t) \\
& + \sum_{k=1}^q \sum_{s=1}^{q'} (-1)^{q+q'+1+s+k} \int_{U \times B} \frac{|w_n|^2}{|a|^2} T_{I(k)J(s), KL}^a \frac{\partial^2 \varphi}{\partial w_{i_k} \partial \bar{w}_{j_s}} d\tau(w, t) \\
& + \sum_{k=1}^q (-1)^{k+q'} \int_{U \times B} \frac{|w_n|^2}{a} T_{I(k)J, KL}^a \frac{\partial^2 \varphi}{\partial w_{i_k} \partial \bar{w}_n} d\tau(w, t) \\
& + \sum_{s=1}^{q'} (-1)^{s+q'} \int_{U \times B} \frac{|w_n|^2}{\bar{a}} T_{IJ(s), KL}^a \frac{\partial^2 \varphi}{\partial w_n \partial \bar{w}_{j_s}} d\tau(w, t) \\
& + \sum_{e=1}^{p-q} \sum_{e'=1}^{p-q'} (-1)^{q+e'+1} \int_{U \times B} \frac{|w_n|^2}{|a|^2} T_{I \cup \{n\} J \cup \{n\}, K_e L_{e'}}^a \frac{\partial^2 \varphi}{\partial t_{i_e} \partial \bar{t}_{j_{e'}}} d\tau(w, t) \\
& + \sum_{e'=1}^{p-q'} (-1)^{e'+1} \int_{U \times B} \frac{|w_n|^2}{a} T_{IJ \cup \{n\}, KL_{e'}}^a \frac{\partial^2 \varphi}{\partial w_n \partial \bar{t}_{j_{e'}}} d\tau(w, t) \\
& + \sum_{e=1}^{p-q} (-1)^{e+q'+1} \int_{U \times B} \frac{|w_n|^2}{\bar{a}} T_{I \cup \{n\} J, K_e L}^a \frac{\partial^2 \varphi}{\partial t_{i_e} \partial \bar{w}_n} d\tau(w, t).
\end{aligned}$$

The lemma 3 gives:

$$\begin{aligned}
& \left| \frac{\partial^2 f_{IJ,KL}}{\partial a \partial \bar{a}}(a) \right| \\
& \leq C_1 \sum_{j=0}^m \frac{\gamma_{dd^c T_j}(|a|)}{|a|^2} + C_2 \sum_{j=0}^m \frac{\gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|)}{|a|^2} \\
& \quad + C_3 \sum_{j=0}^m \frac{\sqrt{\gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|)}}{|a|} \\
& \leq C \left(\sum_{j=0}^m \frac{\gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|)}{|a|^2} + \sum_{j=0}^m \frac{\sqrt{\gamma_{T_j}(|a|) + \gamma_{dd^c T_j}(|a|)}}{|a|} \right) \\
& = C\psi(|a|).
\end{aligned}$$

According to lemma 4, the function $f_{IJ,KL}(a)$ has a limit at 0 if ψ satisfies

$$\int_0^{r_0} r |\log r| \psi(r) dr < +\infty.$$

It's easy to check that there is equivalence between

$$\int_0^{r_0} \sum_{j=0}^m \frac{\gamma_{T_j}(r) + \gamma_{dd^c T_j}(r)}{r} |\log r| dr < +\infty$$

and

$$\int_0^{r_0} \sum_{j=0}^m \frac{\nu_{T_j}(r) - \nu_{T_j}(0)}{r} dr < +\infty \quad \& \quad \int_0^{r_0} \sum_{j=0}^m \frac{\nu_{dd^c T_j}(r)}{r} dr < +\infty.$$

These conditions are exactly the assumptions of the main theorem. So we conclude that

$$(3.2) \quad \int_0^{r_0} \sum_{j=0}^m \frac{\gamma_{T_j}(r) + \gamma_{dd^c T_j}(r)}{r} |\log r| dr < +\infty.$$

Using Cauchy-Schwarz's Inequality, Equation (3.2) gives

$$\begin{aligned}
& \int_0^{r_0} \sqrt{\gamma_{T_j}(r) + \gamma_{dd^c T_j}(r)} |\log r| dr \\
& \leq \left(\int_0^{r_0} \frac{\gamma_{T_j}(r) + \gamma_{dd^c T_j}(r)}{r} |\log r| dr \right)^{1/2} \times \left(\int_0^{r_0} r |\log r| dr \right)^{1/2} < +\infty.
\end{aligned}$$

It follows that:

$$\int_0^{r_0} r |\log r| \psi(r) dr < +\infty$$

This completes the proof of the main theorem.

3.2. Case of closed currents. In the case of positive closed currents, one has a second condition which ensures the existence of the tangent cone, and it is given by the following theorem, where we use the following lemma to show it:

Lemma 5. (See [2]) *Let g be a \mathcal{C}^1 function defined on the pointed disc $\mathbb{D}^*(0, r_0)$. We assume that it exists a measurable function $u :]0, r_0] \rightarrow \mathbb{R}_+$ satisfying $\int_0^{r_0} u(r)dr < +\infty$ such that: $|dg(a)| \leq u(|a|)$. Then $g(a)$ has a limit when a goes to 0.*

Theorem 3. *Let T be a positive closed current of bidegree (p, p) on an open set Ω of $\mathbb{C}^n \times \mathbb{C}^m$. We assume that for all integer $j \in [0, m]$ the current T_j is closed and*

$$\int_0^{r_0} \frac{\sqrt{\nu_{T_j}(r) - \nu_{T_j}(\frac{r}{2})}}{r} dr < +\infty.$$

Then the directional tangent cone to T exists.

Proof. We set

$$\begin{aligned} f_{IJ, KL}(a) &= \int_{U \times B} T_{IJ, KL}^a(w, t) \varphi(w, t) d\tau(w, t) \\ &= \int_{U \times B} T_{IJ, KL}(w', aw_n, t) \varphi(w, t) d\tau(w, t). \end{aligned}$$

By derivation under the integral sign one has:

$$\begin{aligned} \frac{\partial f_{IJ, KL}}{\partial a}(a) &= \int_{U \times B} w_n \frac{\partial T_{IJ, KL}}{\partial w_n}(w', aw_n, t) \varphi(w, t) d\tau(w, t). \\ \frac{\partial f_{IJ, KL}}{\partial \bar{a}}(a) &= \int_{U \times B} \bar{w}_n \frac{\partial T_{IJ, KL}}{\partial \bar{w}_n}(w', aw_n, t) \varphi(w, t) d\tau(w, t). \end{aligned}$$

The coefficient of $dw_{I \cup \{n\}} \wedge d\bar{w}_J \wedge dt_K \wedge d\bar{t}_L$ in dT is given by

$$(-1)^q \frac{\partial T_{IJ, KL}}{\partial w_n} + \sum_{1 \leq k \leq q} (-1)^{k-1} \frac{\partial T_{I(k)J, KL}}{\partial w_{i_k}} + \sum_{1 \leq e \leq p-q} (-1)^{k-1} \frac{\partial T_{I \cup \{n\}J, K_e L}}{\partial t_{i_e}}$$

With $I(k) = (I \setminus \{i_k\} \cup \{n\})$, and $K_e = K \setminus \{i_e\}$.

This coefficient vanishes because $dT = 0$ and one has

$$T_{IJ, KL}(w', aw_n, t) = a^{-1} T_{IJ, KL}^a(w, t).$$

It follows that

$$\begin{aligned} &\frac{\partial T_{IJ, KL}}{\partial w_n}(w', aw_n, t) \\ &= \frac{1}{a} \sum_{1 \leq k \leq q} (-1)^{k+q} \frac{\partial T_{I(k)J, KL}^a}{\partial w_{i_k}} + \frac{1}{a} \sum_{1 \leq e \leq p-q} (-1)^{q+e-1} \frac{\partial T_{I \cup \{n\}J, K_e L}^a}{\partial t_{i_e}} \end{aligned}$$

By substituting this relation in the integral one has $\frac{\partial f_{IJ,KL}}{\partial a}$ and a by integration by parts we obtain:

$$\begin{aligned} \frac{\partial f_{IJ,KL}}{\partial a} &= \frac{1}{a} \sum_{1 \leq k \leq q} (-1)^{k+q-1} \int_{U \times B} w_n T_{I(k)J,KL}^a \frac{\partial \varphi}{\partial w_{i_k}} d\tau(w, t) \\ &\quad + \frac{1}{a} \sum_{1 \leq e \leq p-q} (-1)^{q+e-1} \int_{U \times B} w_n T_{I \cup \{n\}J, K_e L}^a \frac{\partial \varphi}{\partial t_{i_e}} d\tau(w, t) \end{aligned}$$

Naturally we have:

$$\begin{aligned} \frac{\partial f_{IJ,KL}}{\partial \bar{a}} &= \frac{1}{\bar{a}} \sum_{1 \leq l \leq q} (-1)^{l+q-1} \int_{U \times B} \bar{w}_n T_{IJ(l),KL}^a \frac{\partial \varphi}{\partial \bar{w}_{i_l}} d\tau(w, t) \\ &\quad + \frac{1}{\bar{a}} \sum_{1 \leq e' \leq p-q'} (-1)^{q+e'-1} \int_{U \times B} \bar{w}_n T_{I \cup \{n\}J, K_{e'} L}^a \frac{\partial \varphi}{\partial \bar{t}_{j_{e'}}} d\tau(w, t) \end{aligned}$$

The function φ and its derivatives are bounded on $U \times B$. The lemma 3 gives the following estimate

$$\left| \frac{\partial f_{IJ,KL}}{\partial a} \right| + \left| \frac{\partial f_{IJ,KL}}{\partial \bar{a}} \right| \leq C_1 \frac{1}{|a|} \sqrt{\sum_{j=0}^m \gamma_{T_j}(|a|)},$$

According to lemma 5, the function $f_{IJ,KL}$ has a limit at 0 and the directional tangent cone to T exists. \square

Remark 3. The tow conditions of Theorems 2 and 3 are independent as it was shown in [2, rem 3.7].

ACKNOWLEDGEMENT

The authors would like to thank Professors Khalifa Dabbek and Jean-Pierre Demailly for many fruitful discussions concerning this paper.

REFERENCES

- [1] **N. L. Babouche**, Restriction d'un courant positif fermé à une hypersurface complexe, C. R. Acad. Sci. Paris, Ser. I (1993), 751-754.
- [2] **M. Blel, J.-P. Demailly, M. Mozali**, Sur l'existence du cône tangent à un courant positif fermé, Arkiv för Matematik, Volume **28**, Numbers 1-2, (1990), 231-248.
- [3] **J.-P. Demailly**, Complex Analysis and Algebraic Geometry, open book available at <http://www-fourier.ujf-grenoble.fr/demailly/documents.html> (1997).
- [4] **N. Ghiloufi, K. Dabbek**, On the tangent cones to plurisubharmonic currents, preprint arxiv. 1112.3469v1, (2011), 1-18.
- [5] **N. Ghiloufi**, On the Lelong-Demailly numbers of plurisubharmonic currents, C. R. Acad. Sci. Paris, Ser. I (2011), 505-510.
- [6] **S. Giret**, Sur le tranchage et prolongement de courants, Thèse de Doctorat, université de Poitiers (1998), 1-147.
- [7] **F. Haggui**, Existence of tangent cones to plurisubharmonic currents, Complex Variables and Elliptic Equations, Vol. **56**, No. 12 (2011) 1-10.
- [8] **C. O. Kiselman**, Tangents of plurisubharmonic functions, International Symposium in Memory of Hua Loo Keng, vol. II (1988), 157-167.

- [9] **G. Raby**, Tranchage des courants positifs fermés et équation de Lelong-Poincaré, J. Math. pures et appliquées, Vol. **75**, No 3 (1996) 189-209.

E-mail address: `haithem.hawari@yahoo.fr`

E-mail address: `jawhar_x@hotmail.fr`

E-mail address: `noureddine.ghiloufi@fsg.rnu.tn`

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF GABÈS, UNIVERSITY OF GABÈS, ERRIADH CITY 6072 GABÈS TUNISIA.